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Generalized Boltzmann transport theory for relaxational heat conduction

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ABSTRACT

Relaxational heat conduction lacks a modified transport theory at the mesoscopic level. We establish a modified Boltzmann transport theory with the generalized collision term, which can give rise to the convolution relationship between the heat flux and temperature gradient as well as fractional Fourier law. The macroscopic relaxational behaviors are thereafter connected to mesoscopic memory effects in the generalized collision term. The modified Boltzmann transport theory not only provides an underlying explanation for macroscopic relaxational heat conduction but also possesses engineering applications to situations far from equilibrium. The generalized collision term is not unique framework for relaxational heat conduction, and generalizing the drift term in the Boltzmann transport equation (BTE) can also cover macroscopic models. However, this framework will be paired with anomalies in energy continuity and entropy balance, and as a consequence, it is not suggested for relaxational heat conduction.

1. Introduction

Relaxational heat conduction [1-8] is a common non-Fourier class, wherein heat conduction at time $t = t_0$ depends on not only the states at time $t = t_0$ but also the whole history in $[0, t_0]$. One typical class is the theory of heat waves [1], wherein the heat flux $\mathbf{q}(\mathbf{x}, t)$ is formulated as a convolution of the temperature gradient $\nabla T(\mathbf{x}, t)$, namely,

$$\mathbf{q}(\mathbf{x}, t) = -\int_0^t M\left(t - t'\right)\nabla T\left(\mathbf{x}, t'\right)dt', \tag{1}$$

with $M(\xi)$ the relaxation function. Based on Eq. (1), various constitutive models can arise from different choices of $M(\xi)$, and it is therefore considered an universal formulation for wave-like heat conduction. For instance, an exponential kernel, $M(\xi) = \kappa\exp(-\xi/\tau)$, will give rise to the Cattaneo model [6] as follows

$$\mathbf{q} + \tau\frac{\partial\mathbf{q}}{\partial t} = -\kappa\nabla T, \tag{2}$$

where κ denotes the thermal conductivity and τ is the relaxation time. The Cattaneo model is the most celebrated and simple constitutive relation [1] which can overcome infinite speeds of heat propagation traceable to Fourier's law. Another typical relaxation function takes the form of the Dirac delta function, namely, $M(\xi) = \kappa\delta(\xi - \tau)$. This kernel corresponds to the following single-phase-lagging model [7,9],

$$\mathbf{q}(\mathbf{x}, t + \tau) = -\kappa\nabla T(\mathbf{x}, t). \tag{3}$$

which can be connected to the Cattaneo model through the first-order Taylor expansion of the heat flux.

Despite the universality in non-Fourier heat conduction, Eq. (1) cannot cover relaxational behaviors described in terms of fractional-order operators [10-17], i.e., the fractional Fourier law [16] as follows

$$\begin{aligned}\mathbf{q}(\mathbf{x}, t) &= -\kappa\theta^{1-\alpha}\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\nabla T(\mathbf{x}, t) \\ &= -\kappa\theta^{1-\alpha}\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \left(t - t'\right)^{\alpha-1}\nabla T\left(\mathbf{x}, t'\right)dt',\end{aligned} \tag{4}$$

with $\alpha \in (0, 1)$ and θ a material constant. Different from the fractional-order Cattaneo-type models [18-21], the fractional Fourier law is a generalization of Fourier's law rather than the Cattaneo model. The fractional Fourier law also reflects a relaxational behavior between the heat flux and temperature gradient, but it cannot be included by Eq. (1). Another unsatisfactory problem of Eq. (1) is that it is only a macroscopic description yet lacks mesoscopic understandings. However, specific cases of Eq. (1) like the Cattaneo model can be derived from the Boltzmann transport equation (BTE) [22-26] at the mesoscopic level, namely,

$$\frac{\partial f_P}{\partial t} + \mathbf{v}_g \cdot \nabla f_P = C(f_P), \tag{5}$$

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wherein $f_P = f_P(\mathbf{x}, t, \mathbf{k})$ stands for the phonon distribution function, $\mathbf{k}$ denotes the wave vector, $\mathbf{v}_g$ is the phonon group velocity, and $C(f_P)$ stands for the collision term. In the framework of the Boltzmann transport theory, the Cattaneo model emerges from the local equilibrium assumption ($\nabla f_P \approx \nabla f_0$) and single mode relaxation time approximation:

$$C(f_P) = -\frac{f_P - f_0}{\tau}. \tag{6}$$

where $f_0 = \frac{1}{\exp(\hbar\omega/k_B T) - 1}$ is the Bose-Einstein distribution, $\hbar$ is the reduced Planck constant, k_B is the Boltzmann constant, and ω is the angle frequency. Here, the local equilibrium assumption is at the mesoscopic level, while the Cattaneo model describes macroscopic local non-equilibrium effects.

Since a specific case of Eq. (1), the Cattaneo model, possesses a statistical and mesoscopic foundation, it is necessary to establish a fundamental transport theory for Eq. (1). On the other hand, relaxational heat conduction beyond Eq. (1) like the fractional Fourier law calls for transport theory likewise. The main aim of the present paper is to address the two questions. In this work, we propose two subclasses of the phonon BTE which enable the macroscopic relaxation to cover both Eq. (1) and the fractional Fourier law. The first subclass is based on the generalized collision term, while the other generalizing the drift term. We compare the two subclasses, and suggest the generalized collision term as an universal mesoscopic understanding for relaxational heat conduction.

2. Generalized collision term

We first recall the statistical definitions of the heat flux $\mathbf{q}(\mathbf{x}, t)$ and energy density $u(\mathbf{x}, t)$:

$$\mathbf{q}(\mathbf{x}, t) = \int \mathbf{v}_g f_P(\mathbf{x}, t, \mathbf{k})\hbar\omega d\mathbf{k} = \int \mathbf{v}_g [f_P(\mathbf{x}, t, \mathbf{k}) - f_0]\hbar\omega d\mathbf{k}, \tag{7a}$$

$$u(\mathbf{x}, t) = \int f_0 \hbar\omega d\mathbf{k} = \int f_P(\mathbf{x}, t, \mathbf{k})\hbar\omega d\mathbf{k}. \tag{7b}$$

The temporal Laplace transform of an integrable function $\psi(., t)$ is given by

$$\psi_{LT}(., p) = \int_0^{+\infty} \psi(., t)\exp(-pt)dt, \tag{8}$$

wherein p should fulfill $\lim_{t\to+\infty} |\psi(., t)\exp(-t\mathrm{Re}p)| = 0$. In order to obtain Eq. (1), the collision term is generalized as

$$C(f_P) = \int_0^t X(t - t')[f_P(\mathbf{x}, t', \mathbf{k}) - f_0]dt' + \frac{\partial}{\partial t}[f_P(\mathbf{x}, t, \mathbf{k}) - f_0], \tag{9}$$

which satisfies the restriction $\int C(f_P)d\mathbf{k} \equiv 0$. The generalized collision term consists of two terms. The first term reflects the memory effect of the scattering process, namely that the scattering process at t_0 depends on the whole history of the non-equilibrium term $[f_P(\mathbf{x}, t, \mathbf{k}) - f_0]$ in $[0, t_0]$. The second term is contributed by the temporal derivative of the non-equilibrium term at t_0, which does not involve the memory effect. Upon multiplying Eq. (5) by $\mathbf{v}_g\hbar\omega$ and integrating it over the wave vector space, we acquire

$$\frac{\partial \mathbf{q}(\mathbf{x}, t)}{\partial t} + \frac{\kappa}{\tau}\nabla T(\mathbf{x}, t) = \int_0^t X(t - t')\mathbf{q}(\mathbf{x}, t')dt' + \frac{\partial \mathbf{q}(\mathbf{x}, t)}{\partial t}$$
$$\Rightarrow \int_0^t X(t - t')\mathbf{q}(\mathbf{x}, t')dt' = \frac{\kappa}{\tau}\nabla T(\mathbf{x}, t)$$

$$\Rightarrow \mathbf{q}_{LT}(\mathbf{x}, p) = \frac{\kappa}{\tau X_{LT}(p)}\nabla T_{LT}(\mathbf{x}, p). \tag{10}$$

In Eq. (10), the thermal conductivity is calculated as $\kappa = \frac{1}{3}|\mathbf{v}_g|^2 c\tau$, and $c = \int \frac{\partial f_0}{\partial T}\hbar\omega d\mathbf{k}$ is the specific heat capacity per unit volume. The temporal Laplace transform of Eq. (1) reads

$$\mathbf{q}_{LT}(\mathbf{x}, p) = -M_{LT}(p)\nabla T_{LT}(\mathbf{x}, p), \tag{11a}$$

and combining it with Eq. (10) yields

$$X_{LT}(p) = -\frac{\kappa}{\tau M_{LT}(p)} \Rightarrow X(t) = -\frac{\kappa}{2\pi\tau i}\int_{\zeta - i\infty}^{\zeta + i\infty} \frac{\exp(pt)}{M_{LT}(p)}dp, \tag{11b}$$

wherein ζ is an auxiliary parameter to guarantee the convergence. Besides Eq. (1), the generalized collision term expressed by Eqs. (9) and (11b) is able to expect the fractional Fourier law as well, and the corresponding memory kernel is written as

$$X(\xi) = -\frac{\theta^{\alpha-1}}{\Gamma(1-\alpha)\tau\xi^{\alpha}}. \tag{12}$$

Thereupon, we have obtained an universal collision term for relaxational heat conduction expressed by Eqs. (9) and (11b). This generalized collision term provides a mesoscopic understanding for macroscopic phenomenological relaxation based on the Boltzmann transport theory, namely that the evolution of phonon distribution function exhibits memory behaviors.

Our generalized Boltzmann transport theory not only provides an underlying explanation for macroscopic relaxational models but also possesses engineering applications. For the system wherein non-stationary heat conduction is dominated by the memory regime, the non-Fourier relaxation should be considered. For instance, the fractional Fourier law is recently investigated in heat transfer induced by gas adsorption [16]. The experimental results perform a slow thermal diffusion stage, which supports the fractional Fourier law yet cannot be reproduced by Fourier's law. Here, we propose a mesoscopic mechanism for the fractional Fourier law in terms of the following collision term,

$$C(f_P) = -\frac{\theta^{\alpha-1}}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-t')^{\alpha}}\frac{[f_P(\mathbf{x},t',\mathbf{k}) - f_0(\mathbf{x},t')]}{\tau}dt' + \frac{\partial}{\partial t}[f_P(\mathbf{x}, t, \mathbf{k}) - f_0(\mathbf{x}, t)]. \tag{13}$$

As a specific case of Eq. (9), Eq. (13) corresponds to the memory kernel $X(t - t') \propto (t - t')^{-\alpha}$, which implies that the memory effect obeys a power-law decaying. The fractional Fourier law can degenerate to classical Fourier's law, yet always differs from the Cattaneo model, which corresponds to the single mode relaxation time approximation. Thus, Eq. (13) cannot degenerate to the single mode relaxation time approximation. The parameters (α, θ) can be determined by measurements in the near-equilibrium region. Meanwhile, the relaxation time τ can be obtained in a stationary heat conduction process. The one-dimensional heat conduction problem on $[0, L]$ is taken as an example, which is induced by a constant temperature difference ΔT. In this problem, we have $\kappa = -\frac{qL}{\Delta T}$ and the relaxation time is written as $\tau = -\frac{3qL}{\Delta T|\mathbf{v}_g|^2 c}$ with q the one-dimensional heat flux. Indeed, the relaxation time in Eq. (13) corresponds to the stationary transport property, which can only reflect the stationary scattering. In the engineering situations far from equilibrium, neither measurements on macroscopic quantities nor macroscopic constitutive models are accessible. Thereby, one can use a mesoscopic description by Eq. (13) instead.

3. Generalized drift term and comparisons

For a given macroscopic quantity, there exist different phonon distribution functions, and hence the mesoscopic Boltzmann transport theory for macroscopic relaxation is not unique. In the following, we will show that Eq. (1) and the fractional Fourier law can also be obtained via generalizing the drift term. Using the Laplace transform approach stated above, one can demonstrate that the following BTE can also lead to Eq. (1),

$$\frac{\partial f_P(\mathbf{x},t,\mathbf{k})}{\partial t}+\int_0^t \frac{M\left(t-t'\right)}{\kappa}\mathbf{v}_g\cdot\nabla f_P\left(\mathbf{x},t',\mathbf{k}\right)dt' = -\frac{f_P(\mathbf{x},t,\mathbf{k})-f_0(\mathbf{x},t)}{\tau}+\frac{\partial}{\partial t}\left[f_P(\mathbf{x},t,\mathbf{k})-f_0(\mathbf{x},t)\right]. \tag{14a}$$

For the fractional Fourier law, the corresponding BTE is written as

$$\frac{\partial f_P(\mathbf{x},t,\mathbf{k})}{\partial t}+\frac{\theta^{1-\alpha}}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \left(t-t'\right)^{\alpha-1}\mathbf{v}_g\cdot\nabla f_P\left(\mathbf{x},t',\mathbf{k}\right)dt' = -\frac{f_P(\mathbf{x},t,\mathbf{k})-f_0(\mathbf{x},t)}{\tau}+\frac{\partial}{\partial t}\left[f_P(\mathbf{x},t,\mathbf{k})-f_0(\mathbf{x},t)\right]. \tag{14b}$$

Although the generalized drift terms can give rise to macroscopic relaxational models, they will be parried with two anomalies.

The first anomaly is related to the continuity equation. Upon multiplying Eq. (5) by $\hbar\omega$ and integrating it over the wave vector space, we can derive the standard continuity equation,

$$\frac{\partial u(\mathbf{x},t)}{\partial t}+\nabla\cdot\mathbf{q}(\mathbf{x},t)=0. \tag{15}$$

Nevertheless, Eqs. (14a) and (14b) will lead to the following continuity equations, respectively,

$$\frac{\partial u(\mathbf{x},t)}{\partial t}+\nabla\cdot\left[\int_0^t \frac{M\left(t-t'\right)}{\kappa}\mathbf{q}\left(\mathbf{x},t'\right)dt'\right]=0, \tag{16a}$$

$$\frac{\partial u(\mathbf{x},t)}{\partial t}+\nabla\cdot\left[\frac{\theta^{1-\alpha}}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \left(t-t'\right)^{\alpha-1}\mathbf{q}\left(\mathbf{x},t'\right)dt'\right]=0, \tag{16b}$$

which are obviously unconventional. Eq. (16a) will reduce to the standard continuity equation if and only if $M(\xi)=\kappa\delta(\xi)$, while for Eq. (16b), the standard continuity equation occurs only in the limit $\alpha\to 1$. Meanwhile, the constitutive model becomes Fourier's law. It indicates that the constitutive and continuity equations are not independent of each other, and non-Fourier relaxational models must coexist with unconventional continuity equations. We mention that such coexistence has been discussed by previous studies [1,8,27], but the discussion is not based on the generalized drift term. This expectation differs from usual understandings of relaxational heat conduction. For instance, the experimental results in Ref. [16] support a non-Fourier constitutive model and the standard continuity equation.

The other anomaly involves the entropic concepts and entropy balance equation, namely,

$$\frac{\partial s(\mathbf{x},t)}{\partial t}=-\nabla\cdot\mathbf{J}(\mathbf{x},t)+\sigma(\mathbf{x},t), \tag{17}$$

where $s(\mathbf{x},t)$ stands for the local entropy density, $\mathbf{J}(\mathbf{x},t)$ is the entropy flux, and $\sigma(\mathbf{x},t)$ denotes the entropy production rate. In the near-equilibrium region, these entropic concepts can be expressed in the framework of classical irreversible thermodynamics (CIT) [28], namely,

$$\begin{cases} s(\mathbf{x},t)=\int^{T(\mathbf{x},t)}\frac{c}{T}dT \\ \mathbf{J}(\mathbf{x},t)=\frac{\mathbf{q}(\mathbf{x},t)}{T(\mathbf{x},t)} \\ \sigma(\mathbf{x},t)=\mathbf{q}(\mathbf{x},t)\cdot\nabla\left[\frac{1}{T(\mathbf{x},t)}\right] \end{cases}. \tag{18}$$

At the statistical level, the CIT formulation can be obtained from Boltzmann-Gibbs statistical mechanics and Eq. (5), which will be subsequently illustrated. In Boltzmann-Gibbs statistical mechanics, the entropy density of phonons is written as

$$s=k_B\int\left[(f_P+1)\ln(f_P+1)-f_P\ln f_P\right]d\mathbf{k}, \tag{19}$$

whose temporal derivative reads

$$\frac{\partial s}{\partial t}=k_B\int\frac{\partial f_P}{\partial t}\ln\left(\frac{f_P+1}{f_P}\right)d\mathbf{k}. \tag{20}$$

Substituting Eq. (5) into Eq. (20) yields

$$\frac{\partial s}{\partial t}=-\nabla\cdot\left\{\int\mathbf{v}_g k_B\left[(f_P+1)\ln(f_P+1)-f_P\ln f_P\right]d\mathbf{k}\right\} +k_B\int C(f_P)\ln\left(\frac{f_P+1}{f_P}\right)d\mathbf{k}, \tag{21}$$

and we thereafter obtain

$$\mathbf{J}=\int\mathbf{v}_g k_B\left[(f_P+1)\ln(f_P+1)-f_P\ln f_P\right]d\mathbf{k}, \tag{22a}$$

$$\sigma=k_B\int C(f_P)\ln\left(\frac{f_P+1}{f_P}\right)d\mathbf{k}. \tag{22b}$$

When the distribution function is sufficiently close to the equilibrium distribution, we can employ the following expansions,

$$\ln\left(\frac{f_P+1}{f_P}\right)=\ln\frac{f_0+1}{f_0}-\frac{(f_P-f_0)}{(f_0+1)f_0}+O\left(|f_P-f_0|^2\right) =\frac{\hbar\omega}{k_BT}-\frac{(f_P-f_0)}{(f_0+1)f_0}+O\left(|f_P-f_0|^2\right), \tag{23a}$$

$$(f_P+1)\ln(f_P+1)-f_P\ln f_P=(f_0+1)\ln(f_0+1)-f_0\ln f_0 +(f_P-f_0)\ln\left(\frac{f_0+1}{f_0}\right)+O\left(|f_P-f_0|^2\right). \tag{23b}$$

Using Eq. (23a), the entropy density in Eq. (19) can be simplified as

$$\frac{\partial s}{\partial T}=k_B\int\frac{\partial f_P}{\partial T}\ln\left(\frac{f_P+1}{f_P}\right)d\mathbf{k} =k_B\int\frac{\partial f_P}{\partial T}\left[\frac{\hbar\omega}{k_BT}+O(|f_P-f_0|)\right]d\mathbf{k} =\frac{c}{T}+k_B\int\frac{\partial f_P}{\partial T}O(|f_P-f_0|)d\mathbf{k}, \tag{24}$$

which agrees with the CIT entropy density as the remainder term is neglected. Upon combining Eq. (23b) and $\int\mathbf{v}_g k_B[(f_0+1)\ln(f_0+1)-f_0\ln f_0]d\mathbf{k}\equiv\mathbf{0}$, Eq. (22a) is reformed as

$$\mathbf{J}=\int\mathbf{v}_g k_B\left[(f_P-f_0)\ln\left(\frac{f_0+1}{f_0}\right)+O\left(|f_P-f_0|^2\right)\right]d\mathbf{k} =\int\mathbf{v}_g k_B\left[(f_P-f_0)\frac{\hbar\omega}{k_BT}+O\left(|f_P-f_0|^2\right)\right]d\mathbf{k} =\frac{\mathbf{q}}{T}+\int\mathbf{v}_g k_B O\left(|f_P-f_0|^2\right)d\mathbf{k}. \tag{25}$$

With the remainder term neglected, the CIT entropy flux is reproduced. Owing to the entropy balance equation, the entropy production rate in Eq. (18) should also coincide with the CIT framework.

Accordingly, the entropic concepts based on Eq. (5) are consistent with the CIT framework in the near-equilibrium region. Nevertheless, Eqs. (14a) and (14b) will expect entropic concepts deviating from the CIT framework even in the absence of the remainder terms. Eqs. (14a) and (14b) correspond to the following entropy balance equations, respectively,

$$\frac{\partial s}{\partial t} = -k_B \int \left[\int_0^t Y(t-t')\mathbf{v}_g \cdot \nabla f_P(\mathbf{x}, t', \mathbf{k}) dt' \right] \ln\left(\frac{f_P+1}{f_P}\right) d\mathbf{k} + k_B \int C(f_P) \ln\left(\frac{f_P+1}{f_P}\right) d\mathbf{k}, \tag{26a}$$

$$\frac{\partial s}{\partial t} = -k_B \int \left[\frac{\theta^{1-\alpha}}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-t')^{\alpha-1}\mathbf{v}_g \cdot \nabla f_P(\mathbf{x}, t', \mathbf{k}) dt' \right] \ln\left(\frac{f_P+1}{f_P}\right) d\mathbf{k} + k_B \int C(f_P) \ln\left(\frac{f_P+1}{f_P}\right) d\mathbf{k}, \tag{26b}$$

Eqs. (26a) and (26b) imply that there exist no explicit expressions for the entropy flux, while the implicit expressions are respectively given by

$$\nabla \cdot \mathbf{J}(\mathbf{x}, t) = k_B \int \left[\int_0^t Y(t-t')\mathbf{v}_g \cdot \nabla f_P(\mathbf{x}, t', \mathbf{k}) dt' \right] \ln\left(\frac{f_P+1}{f_P}\right) d\mathbf{k}, \tag{27a}$$

$$\nabla \cdot \mathbf{J}(\mathbf{x}, t) = k_B \int \left[\frac{\theta^{1-\alpha}}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-t')^{\alpha-1}\mathbf{v}_g \cdot \nabla f_P(\mathbf{x}, t', \mathbf{k}) dt' \right] \ln\left(\frac{f_P+1}{f_P}\right) d\mathbf{k}. \tag{27b}$$

Using the expansion method stated above and neglecting the remainder terms., Eqs. (27a) and (27b) can be respectively approximated as

$$\mathbf{J}(\mathbf{x}, t) \approx \frac{1}{T(\mathbf{x}, t)} \left[\int_0^t \frac{M(t-t')}{\kappa} \mathbf{q}(\mathbf{x}, t') dt' \right], \tag{28a}$$

$$\mathbf{J}(\mathbf{x}, t) \approx \frac{1}{T(\mathbf{x}, t)} \frac{\theta^{1-\alpha}}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-t')^{\alpha-1} \mathbf{q}(\mathbf{x}, t') dt'. \tag{28b}$$

The corresponding approximations of the entropy production rate are thereafter calculated as follows, respectively,

$$\sigma(\mathbf{x}, t) = \left[\int_0^t \frac{M(t-t')}{\kappa} \mathbf{q}(\mathbf{x}, t') dt' \right] \cdot \nabla \left[\frac{1}{T(\mathbf{x}, t)} \right], \tag{29a}$$

$$\sigma(\mathbf{x}, t) = \nabla \left[\frac{1}{T(\mathbf{x}, t)} \right] \cdot \left[\frac{\theta^{1-\alpha}}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-t')^{\alpha-1} \mathbf{q}(\mathbf{x}, t') dt' \right]. \tag{29b}$$

Obviously, the above entropic functionals deviate from the CIT framework even in the near-equilibrium region.

In order to show the influence of anomalies in the continuity and entropy balance equations quantitatively, we consider the single-phase-lagging model as a comparative example. For this model, the generalized drift term gives rise to the following continuity equation,

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} + \nabla \cdot \mathbf{q}(\mathbf{x}, t+\tau) = 0, \tag{30a}$$

and meanwhile, the entropy flux and entropy production rate are formulated as

$$\begin{cases} \mathbf{J}(\mathbf{x}, t) = \frac{\mathbf{q}(\mathbf{x}, t+\tau)}{T(\mathbf{x}, t)} \\ \sigma(\mathbf{x}, t) = \mathbf{q}(\mathbf{x}, t+\tau) \cdot \nabla\left[\frac{1}{T(\mathbf{x}, t)}\right] \end{cases}. \tag{30b}$$

It is observed that the differences between the anomalous and standard cases are traceable to the difference between $\mathbf{q}(\mathbf{x}, t+\tau)$ and $\mathbf{q}(\mathbf{x}, t)$. Through employing the first-order expansion, we can acquire the quantitative estimations, respectively,

$$\nabla \cdot [\mathbf{q}(\mathbf{x}, t+\tau) - \mathbf{q}(\mathbf{x}, t)] \approx \tau \nabla \cdot \left[\frac{\partial \mathbf{q}(\mathbf{x}, t)}{\partial t} \right], \tag{31a}$$

$$\begin{cases} \frac{\mathbf{q}(\mathbf{x}, t+\tau)}{T(\mathbf{x}, t)} - \frac{\mathbf{q}(\mathbf{x}, t)}{T(\mathbf{x}, t)} \approx \frac{\tau}{T(\mathbf{x}, t)} \frac{\partial \mathbf{q}(\mathbf{x}, t)}{\partial t} \\ [\mathbf{q}(\mathbf{x}, t+\tau) - \mathbf{q}(\mathbf{x}, t)] \cdot \nabla\left[\frac{1}{T(\mathbf{x}, t)}\right] \approx \tau \frac{\partial \mathbf{q}(\mathbf{x}, t)}{\partial t} \cdot \nabla\left[\frac{1}{T(\mathbf{x}, t)}\right] \end{cases}, \tag{31b}$$

which indicate that the influence of anomalies in the continuity and entropy balance equations depends on $|\frac{\partial \mathbf{q}(\mathbf{x}, t)}{\partial t}|$.

To sum up, the validity of the generalized drift term is debatable because it will paired with anomalies in energy continuity and entropy balance. Accordingly, we suggest the generalized collision term rather than the generalized drift term as the modified transport theory for relaxational heat conduction.

4. Summary

A modified Boltzmann transport theory is established for relaxational heat conduction, which can give rise to the convolution relationship between the heat flux and temperature gradient as well as fractional Fourier law, which is the original content different from previous studies. The main idea of this work is establishing such transport theory via generalizing the collision term. The generalized collision term not only provides an underlying explanation for macroscopic relaxational heat conduction but also possesses engineering applications to situations far from equilibrium. In framework of this approach, the energy continuity equation and entropy concepts obey existing theories.

The Boltzmann transport theory for relaxational heat conduction is not unique, which can also be achieved through generalizing the drift term. Different from the generalized collision term, the generalized drift term will give rise to the unconventional continuity equations and entropic concepts. To the best of our knowledge, there exists no experiment which can support these anomalous behaviors. Thus, we suggest the Boltzmann transport theory with the generalized collision term as the mesoscopic description for relaxational heat conduction.

Declaration of Competing Interest

There are no conflicts to declare.

CRediT authorship contribution statement

Shu-Nan Li: Formal analysis, Writing - original draft. **Bing-Yang Cao:** Writing - review & editing.

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References

[1] D.D. Joseph, L. Preziosi, Heat waves, Rev. Mod. Phys. 61 (1989) 41–73.
[2] Z.Y. Guo, Energy-mass duality of heat and its applications, ES Energy & Environment 1 (2018) 4–15.
[3] M.T. Xu, A non-local constitutive model for nano-scale heat conduction, Int. J. Thermal Sci. 134 (2018) 594–600.
[4] G. Chen, Ballistic-diffusive heat-conduction equations, Phys. Rev. Lett. 86 (2001) 2297–2300.
[5] Y.Y. Guo, M.R. Wang, Phonon hydrodynamics and its applications in nanoscale heat transport, Phys. Rep. 595 (2015) 1–44.

[6] C. Cattaneo, Sur une forme de léquation de lachaleur éliminant le paradoxe d'une propagation instantanée, Comptes Rendus 247 (1958) 431–433.
[7] S.L. Sobolev, Equations of transfer in non-local media, Int. J. Heat Mass Transfer 37 (1994) 2175–2182.
[8] S.L. Sobolev, Transport processes and traveling waves in systems with local nonequilibrium, Sov. Phys. Usp. 34 (1991) 217–229.
[9] L. Cheng, M.T. Xu, L.Q. Wang, From Boltzmann transport equation to single-phase-lagging heat conduction, Int. J. Heat Mass Transfer 51 (2008) 6018–6023.
[10] J. Hristov, Transient heat diffusion with a non-singular fading memory: From the Cattaneo constitutive equation with Jeffrey's kernel to the Caputo-Fabrizio time-fractional derivative, Thermal Sci 20 (2016) 757–762.
[11] J. Hristov, Steady-state heat conduction in a medium with spatial non-singular fading memory: Derivation of Caputo-Fabrizio space-fractional derivative with Jeffrey's kernel and analytical solutions, Thermal Sci 21 (2017) 827–839.
[12] M. Zingales, Fractional-order theory of heat transport in rigid bodies, Commun. Nonlinear Sci. Numer. Simulat. 19 (2014) 3938–3953.
[13] M.S. Mongioví, M. Zingales, A non-local model of thermal energy transport: The fractional temperature equation, Int. J. Heat Mass Transfer 67 (2013) 593–601.
[14] F. Falcini, R. Garra, V. Voller, Modeling anomalous heat diffusion: Comparing fractional derivative and non-linear diffusivity treatments, Int. J. Therm. Sci. 137 (2019) 584–588.
[15] A. Kundu, C. Bernardin, K. Saito, A. Kundu, A. Dhar, Fractional equation description of an open anomalous heat conduction set-up, J. Stat. Mech. Theory Exp. 1 (2019) 013205.
[16] J.H. Kang, D. Zhang, F.B. Zhou, H.J. Li, T.Q. Xia, Numerical modeling and experimental validation of fractional heat transfer induced by gas adsorption in heterogeneous coal matrix, Int. J. Heat Mass Transfer 128 (2019) 492–503.
[17] S.N. Li, B.Y. Cao, Anomalous heat diffusion from fractional Fokker-Planck equation, Appl. Math. Lett. 99 (2020) 105992.
[18] A. Compte, R. Metzler, The generalized Cattaneo equation for the description of anomalous transport processes, J. Phys. A: Math. Gen. 30 (1997) 7277–7289.
[19] R. Metzler, A. Compte, Stochastic foundation of normal and anomalous Cattaneo-type transport, Physica A 268 (1999) 454–468.
[20] V. Zaburdaev, S. Denisov, J. Klafter, Lévy walks, Rev Mod. Phys. 87 (2015) 483–530.
[21] T. Atanacković, S. Konjik, L. Oparnica, D. Zorica, The Cattaneo type space-time fractional heat conduction equation, Continuum Mech. Thermodyn. 24 (2012) 293–311.
[22] M.T. Xu, X.F. Li, The modeling of nanoscale heat conduction by Boltzmann transport equation, Int. J. Heat Mass Transfer 55 (2012) 1905–1910.
[23] S.N. Li, B.Y. Cao, Fractional Boltzmann transport equation for anomalous heat transport and divergent thermal conductivity, Int. J. Heat Mass Transfer 137 (2019) 84–89.
[24] S.N. Li, B.Y. Cao, Fractional-order heat conduction models from generalized Boltzmann transport equation, Phil. Trans. R. Soc. A 378 (2020) 20190280.
[25] T. Xue, X. Zhang, Kumar K. Tamma, Generalized heat conduction model in moving media emanating from Boltzmann transport equation, Int. J. Heat Mass Transfer 119 (2018) 148–151.
[26] K. Razi-Naqvi, S. Waldenstrøm, Brownian motion description of heat conduction by phonons, Phys. Rev. Lett. 95 (2005) 065901.
[27] S.L. Sobolev, Space-time nonlocal model for heat conduction, Phys. Rev. E 50 (1994) 3255–3258.
[28] D. Jou, J. Casas-Vazquez, G. Lebon, Extended Irreversible Thermodynamics, Springer, Heidelberg, 2010.